14. System Optimization

NASA ESMD Capstone Design

developed by

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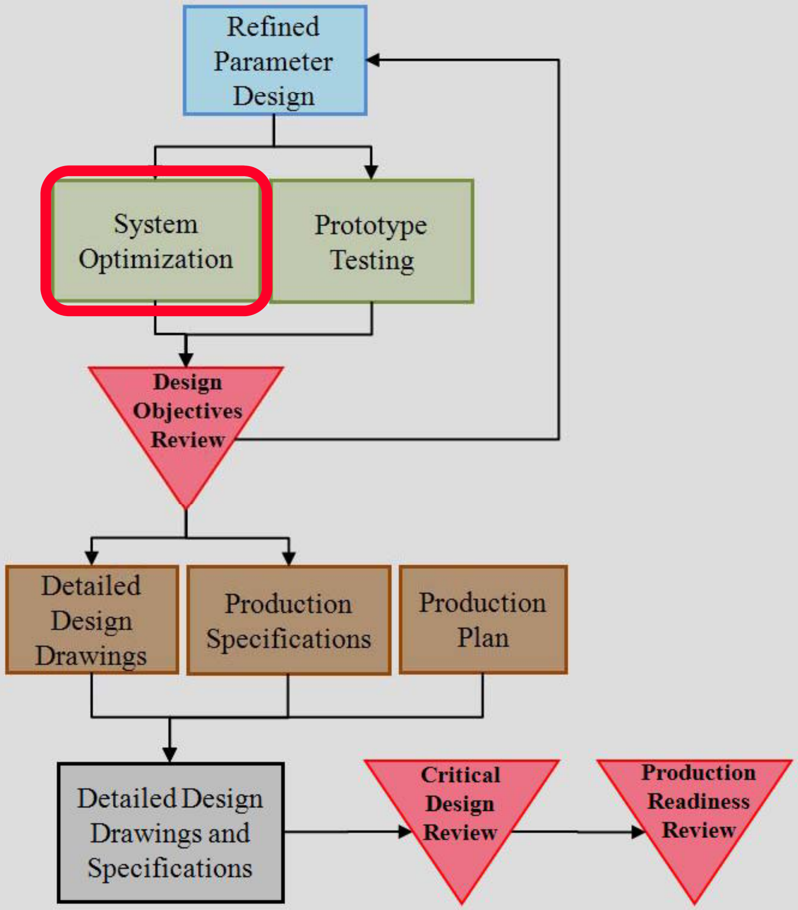
Where in the Process?



Phase C: Optimized Parameter Design

Project Tools

- System Optimization
  - Maximize a desired quality or minimize an undesired one to obtain the best design.

Optimization Methods in Design

- By evolution
  - Often used in the past by improving upon existing designs
- By intuition
  - The art of engineering by intuition is knowing what to do without knowing why - subconsciously - and intuition (creative thinking) has had (and still has) an important role in technological development

Optimization Methods in Design

- Trial-and-error modeling
  - This “guesswork” cannot be called optimization
- By numerical algorithm
  - Linear programming is the most widely applied among a number of techniques; however, most design problems in mechanical engineering are non-linear

(Lumsdaine *et al.*, 2006)

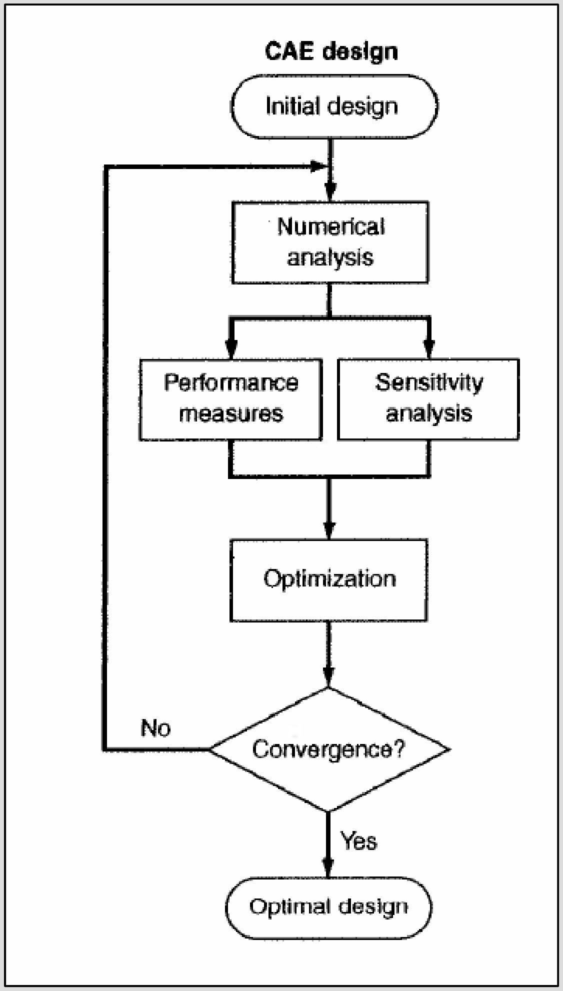
Optimization Methods in Design

- Differential calculus
  – As discussed in today's lecture
- Other means to achieve optimal quality (e.g., Taguchi methods)

Optimization

- Generally there is more than one solution to a design problem, and the first solution is not necessarily the best
  - Thus, the need for optimization is inherent in the design process
- A mathematical theory of optimization has become highly developed and is being applied to design where design functions can be expressed by mathematical equations or with finite-element computer modeling

Design Optimization



(Lumsdaine *et al.*, 2006)

Optimization Problem

- Objective
  - Goal (Minimization, Maximization)
- Design variables and limits
- Constraints
  - Limits in which objective must meet

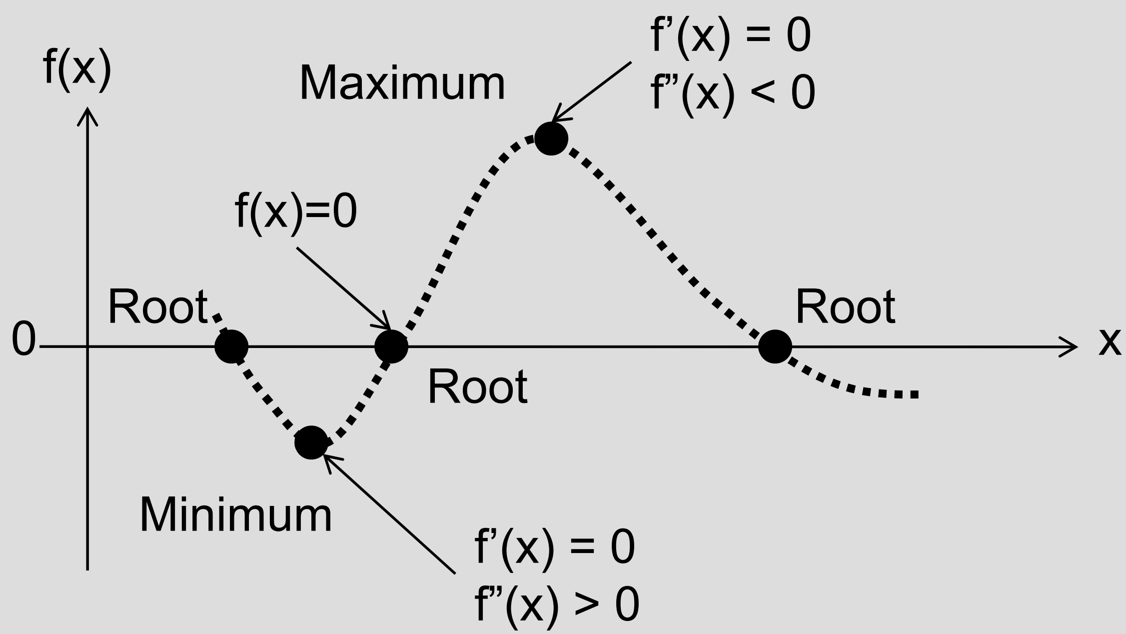
Approaches

- Functional (calculus)
  - Derive a objective function in terms of variables
  - Set derivatives equal to zero
  - Check for local maxima and local minima
- Graphical
  - Plot the objective function in terms of two variables (hold others constant)
  - Find local maxima and local minima

Approaches

- Numerical
  - Define objective function in terms of variables
  - Define constraint functions
  - Solve using Gradient Methods (steepest ascent/descent), Linear or Quadratic Programming, Branch and Bound (discrete optimization)
- Experimental
  - If no explicit relationship exists among variables, use design of Experiments (DOE) or Regression Analysis to determine important parameters
  - Create response surface (second order curve) or linear regression to describe relationship among parameters
  - Use functional, graphical, or numerical means to solve

Functional



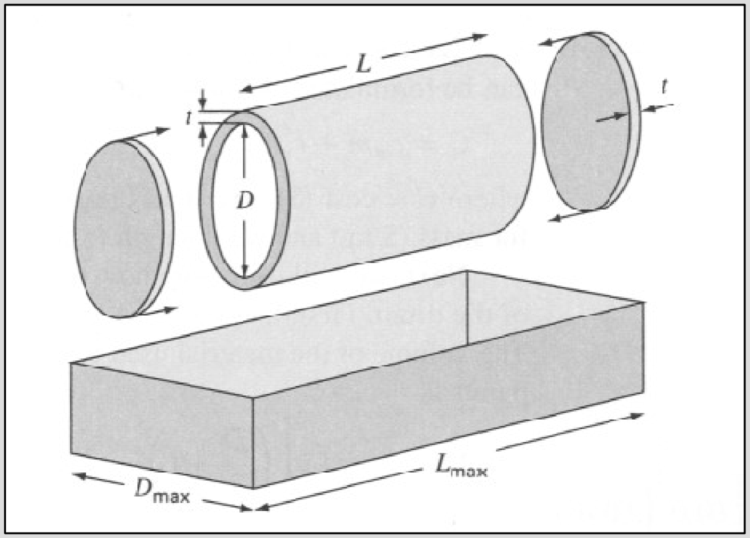
- Roots : Searching for zeros
- Optimization : Searching for maximum or minimum

Engineering Examples

- Design aircraft for minimum weight and maximum strength
- Design pump and heat transfer equipment for maximum efficiency
- Statistical analysis and models with minimum error
- Minimize waiting and idling times
- Shortest route of salesperson visiting various cities during one sales trip
- Machining strategy for minimum cost

Example: Least-Cost Design of Tank

- <u>Problem Statement</u>
  Design a small tank to transport toxic waste to be mounted on the back of a pickup truck



Parameter	Symbol	Value	Units
Required volume	V_o	0.8	m^3
Thickness	t	3	cm
Density	ρ	8000	kg/m^3
Bed length	L_{max}	2	m
Bed width	D_{max}	1	m
Material cost	c_m	4.5	\$/kg
Welding cost	c_w	20	\$/m

Problem Formulation

Objective: Minimize Cost of Tank

Cost of Tank
$$C = \underbrace{c_m m}_{\text{Material Expense}} + \underbrace{c_w l_w}_{\text{Welding Expense}}$$

Constraints:

1. Fit within the truck bed

$$L \leq L_{\max} \qquad D \leq D_{\max}$$

1. Carry required volume of waste

$$\frac{\pi D^2 L}{4} = V_0$$

Problem Formulation

where

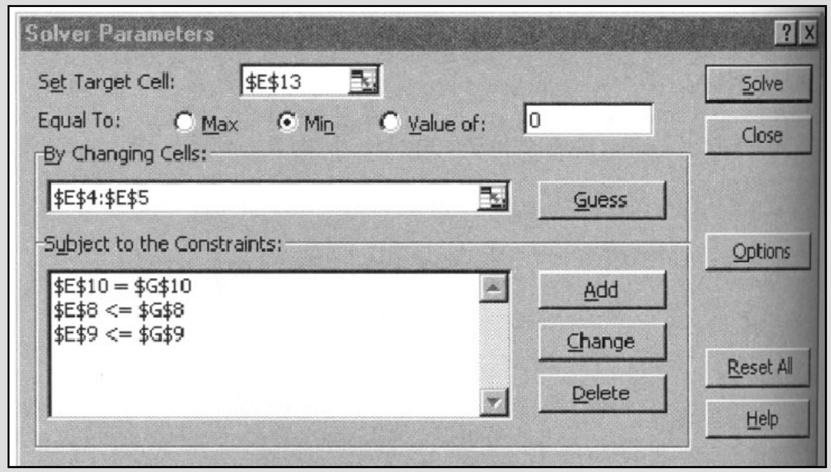
$$m = \rho \left\{ \underbrace{L\pi \left[\left(\frac{D}{2} + t \right)^2 - \left(\frac{D}{2} \right)^2 \right]}_{\text{Volume of Cylinder}} + \underbrace{2\pi \left(\frac{D}{2} + t \right)^2 t}_{\text{Volume of Plate}} \right\}$$

$$l_w = 2\left[2\pi \left(\frac{D}{2} + t \right) + 2\pi \left(\frac{D}{2} \right) \right] = 4\pi (D + t)$$

Solution

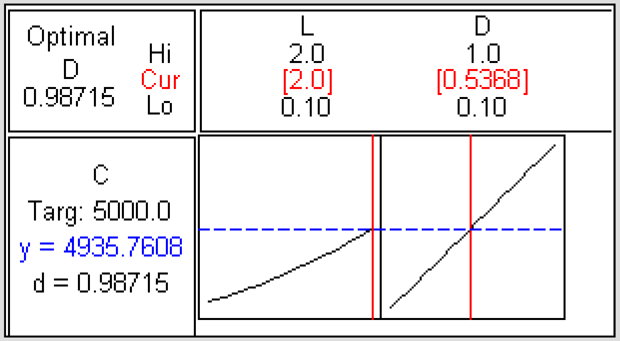
- Numerically: use Excel solver or Matlab optimization toolbox

	A	B	C	D	E	F	G
1	**Optimum tank design**						
2							
3	**Parameters:**			**Design variables:**			
4	V0	0.8		D	0.98351		
5	t	0.03		L	1.053033		
6	rho	8000					
7	Lmax	2		**Constraints:**			
8	Dmax	1		D	**0.98351**	<=	1
9	cm	4.5		L	**1.053033**	<=	2
10	cw	20		Vol	**0.799999**	=	[illegible]
11							
12	**Computed values:**			**Objective function:**			
13	m	**1215.206**		C	**5723.149**		
14	lw	**12.73614**					
15							
16	Vshell	**0.100587**					
17	Vends	**0.051314**					

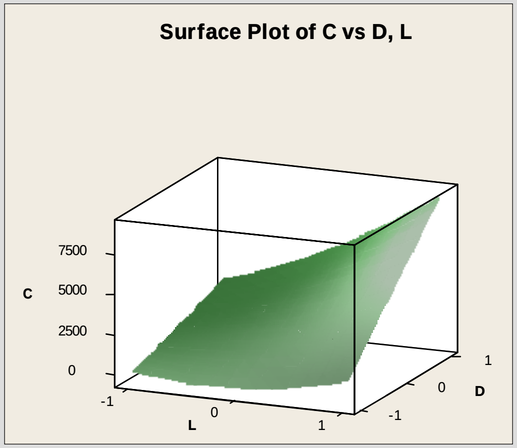


Solution

- Experimentally
  - Construct 2 level factorial design DOE with center points (MINITAB)
  - Create FE or solid model for each **L** and **D** combination to calculate mass and volume
  - Create response surface of costs from data and estimate cost from response (done using linear regression)
  - Check to make sure constraints are met, graphically
  - Works great if no equations are known relating objective function and design parameters



L	D	m	Lw	C
0.1	0.1	19.45274	1.633628	120.2099
2	0.1	205.6864	1.633628	958.2612
0.1	1	501.2474	12.94336	2514.48
2	1	1976.791	12.94336	9154.425
1	0.5	517.835	6.660176	2463.461

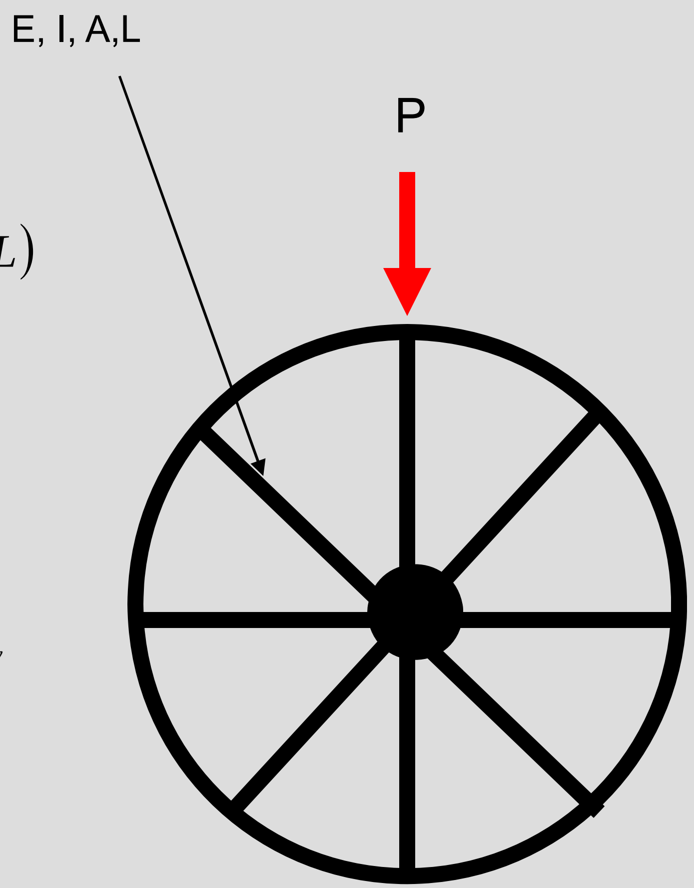


Example: Wheel Design

- Objective minimize weight of wheel of given diameter

$$\min(W = \rho AL)$$

- Constraints
  - No buckling failure $\frac{\pi^2 EI}{AL^2} \le \sigma_y$
  - No bending failure $\frac{PLx/2}{I} \le \sigma_y$
  - No axial failure $\frac{P}{A} \le \sigma_y$
  - In either in-plane or out-of-plane direction



Square Spoke Cross-section

Matlab Code

```
% NLP Optimization of Wheel
% Objective function

function [f] = obj_fun(x)

% x(1) = base (in)
% x(2) = height (in)
% f = weight of arm (lb)

% Call global variables
global p P L E Smax

A = x(1)*x(2); % cross sectional area (in^2)

f = L*A*p; % weight (lb)
% NLP Optimization of Wheel
% Nonlinear constraints

% x(1) = base (in)
% x(2) = height (in)
% c = inequality constraint
% ceq = equality constraint

function [c, ceq] = nlc(x)

% Call global variables
global p P E L Smax

A = x(1)*x(2); % cross sectional area (in^2)
Ib = 1/12*x(1)*(x(2))^3; % area moment of inertia (in^4)
Ih = 1/12*x(2)*(x(1))^3; % area moment of inertia (in^4)
S1 = P/A; % internal stress (psi)
S2 = (P*L*x(2)/2)/Ib; % bending stress
S3 = (P*L*x(1)/2)/Ih; % bending stress
S4 = (pi^2*E*Ih)/(A*L^2); % critical Euler buckling stress (psi)
S5 = (pi^2*E*Ib)/(A*L^2); % critical Euler buckling stress (psi)

% Set constraints
c(1) = S1 - Smax;
c(2) = S2 - Smax;
c(3) = S3 - Smax;
c(4) = S3 - Smax/2;
c(5) = S3 - Smax/2;
ceq = [];
% NLP Optimization of Wheel
% Nonlinear constraints

% x(1) = base (in)
% x(2) = height (in)
% f = weight of arm (lb)

% Define global variables
global p P L E Smax
p = 0.28; % density (lb/in^3)
P = 25; % external load (lb)
E = 27.5e6; % elastic modulus (psi)
Smax = 75e3; % max stress (psi)
L = .75; % length (m)
% NLP minimize obj_fun subject to nonlinconst
x0 = [0.1 0.1];
LB = [0 0];
UB = [0.5 0.5];

x = fmincon('obj_fun',x0,[],[],[],[],LB,UB,'nlc');
f = obj_fun(x);

% Print output
disp(['Optimal base = ' num2str(x(1))])
disp(['Optimal height = ' num2str(x(2))])
disp(['Minimum weight = ' num2str(f)]);
```

Sensitivity Analysis

- Find design parameters most critical to performance of design
- If functional relationship exists – find most sensitive parameters
- Taught in fluid dynamics

$$C^{opt} = 2\sqrt{K_1 K_2 Q} + K_3 Q$$

$$\frac{\partial C^{opt}}{\partial K_2} = \sqrt{\frac{K_1 Q}{K_2}} \qquad \frac{\partial C^{opt}}{\partial K_1} = \sqrt{\frac{K_2 Q}{K_1}} \qquad \frac{\partial C^{opt}}{\partial Q} = \sqrt{\frac{K_1 K_2}{Q}} + K_3 \qquad \frac{\partial C^{opt}}{\partial K_3} = Q$$

Sensitivity Analysis

- Relative Sensitivity

$$S_{K_1}^{C} = \frac{\partial C^{opt} / C^{opt}}{\partial K_1 / K_1} = \frac{\partial \ln C^{opt}}{\partial \ln K_1} = \sqrt{\frac{K_2 Q}{K_1}} \cdot \frac{K_1}{C^{opt}}$$

- If no analytical relationship, can use numerical comparison like changing each parameter by small amount (5%, 10%, *etc.*) using a spreadsheet

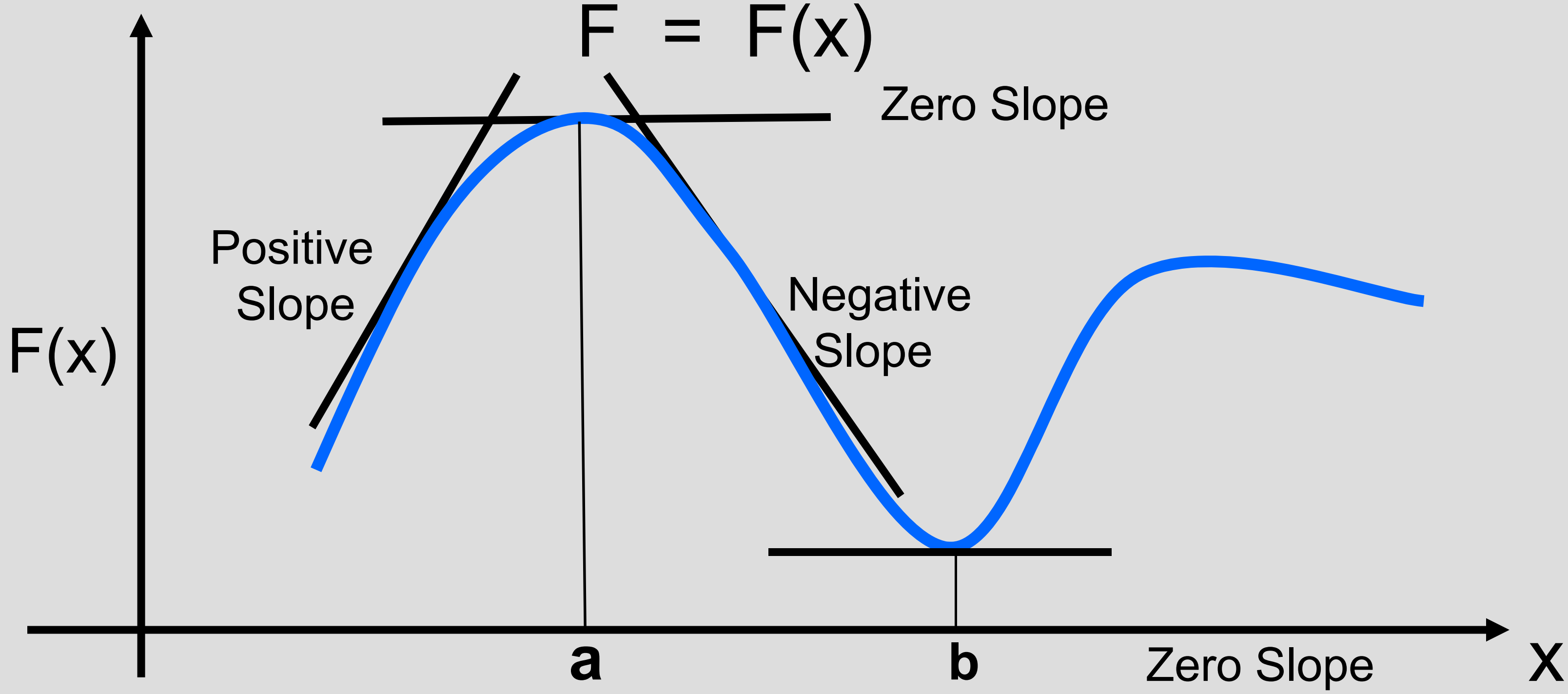
Pi Theorem and Similitude

- Principle of Dimensional Homogeneity
  - Each additive term have same dimensions (*e.g.*, terms of Bernoulli's equation)
- Reduce *n* dimensional variables into *k* dimensionless variables or Π's
- Use Pi Theorem to scale to experiments (similitude) either dynamically (same force ratios) or geometrically (same linear dimension ratios)

How to Determine Maxima and Minima

- Consider a function F for a single variable x, so that

$$F = F(x)$$



How to Determine Maxima and Minima

- When the slope is zero, we see that there is a maximum or a minimum

$$\frac{dF(b)}{dx} = \frac{dF(a)}{dx} = 0 \quad (\text{max} \quad \text{or} \quad \text{min})$$

- To find whether we have a max or min, we can use the Taylor's series expansion and show that

$$\frac{d^2F(a)}{dx^2} < 0 \qquad \text{max}$$

$$\frac{d^2F(b)}{dx^2} > 0 \qquad \text{min}$$

Example 1: Tank (Single Variable)

Design a cylindrical tank to store a fixed volume of liquid. The tank will be constructed by forming and welding thin steel plate. Therefore, the primary cost will depend on the area A of steel plate that is used.

D = tank diameter, h = height

C = cost/unit area, V = volume of liquid

$$A = 2\pi r^2 + 2\pi r h \quad \text{or} \quad A = 2(\pi D^2/4) + \pi Dh$$

$$U \text{ (total cost)} = CA = C\,[D^2 \pi/2 + \pi Dh]$$

U is also known as the objective function

Example 1 (continued)

A functional constraint is that the tank must hold a specified volume:

$$V = \pi r^2 h = \pi D^2 h/4$$

To find the optimum diameter (minimum cost):

$$\frac{dU}{dD} = C\left[\pi D + \pi \frac{dh}{dD}\right] = C\left[\pi D - 4V/D^2\right] = 0$$

Therefore, $D = (4V/\pi)^{1/3} = 1.084\ V^{1/3}$

Example 1 (continued)

To determine whether this diameter will produce max or min cost, we take the second derivative:

$$\frac{d^2U}{dD^2} = C \left[\pi - 4 \frac{d}{dD} \left(\frac{V}{D^2} \right) \right]$$

$$= C \left[\pi - \frac{0 - 4V\,(2D)}{D^4} \right]$$

$$= C \left[\pi + 8\,V/D^2 \right] > 0$$

Therefore, the calculated diameter D will produce minimum cost (for the fixed volume)

Finding Optima in Multivariable Problems

- Example 1 shows how to determine the optimum for a single dependent variable D
- The Lagrange multipliers are a powerful method for finding optima in multivariable problems involving constraints
- The objective function

$$U=U(x,y,z)$$

is now subject to constraint

$$\Phi(x,y,z)=0$$

Finding Optima in Multivariable Problems (continued)

The Lagrange expression becomes

$$\text{Lagrangian} = U(x,y,z) + \lambda\, \varphi(x,y,z) = 0$$

where λ is the Lagrange multiplier. In the literature it is known as the "method of undetermined multipliers."

If the function U is to have a maximum or minimum, then

$$\frac{\partial U}{\partial x}dx = 0, \frac{\partial U}{\partial y}dy = 0, \frac{\partial U}{\partial z}dz = 0$$

Finding Optima in Multivariable Problems (continued)

Hence,

$$\frac{\partial U}{\partial x} = 0, \frac{\partial U}{\partial y} = 0, \frac{\partial U}{\partial z} = 0$$

or,

$$\frac{\partial U}{\partial x} dx = 0, \frac{\partial U}{\partial y} dy = 0, \frac{\partial U}{\partial z} dz = 0$$

Since, $\varphi(x, y, z) = 0$

The total derivative(chain rule):

$$d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz = 0$$

Finding Optima in Multivariable Problems (continued)

When adding Eq. (1) and Eq. (2), we have

$$(\frac{\partial U}{\partial x} + \lambda \frac{\partial \varphi}{\partial x})dx + (\frac{\partial U}{\partial y} + \lambda \frac{\partial \varphi}{\partial y})dy + (\frac{\partial U}{\partial z} + \lambda \frac{\partial \varphi}{\partial z})dz = 0 \qquad (3)$$

Eq. (3) is satisfied if

$$\frac{\partial U}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0$$

$$\frac{\partial U}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0$$

$$\frac{\partial U}{\partial z} + \lambda \frac{\partial \varphi}{\partial z} = 0$$

Example 2: Largest Volume

Find the volume of the largest parallelepiped that can be inscribed in the ellipsoid

$\varphi(x,y,z) = x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 = 0$ *(constraint)* (a)

$U(x,y,z) = 8xyz$ *(function to be maximized)* (b)

$$\frac{\partial U}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0 \qquad \frac{\partial U}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0 \qquad \frac{\partial U}{\partial z} + \lambda \frac{\partial \varphi}{\partial z} = 0 \qquad \text{(c)}$$

Example 2 (continued)

Applying the Lagrange equation (c) to Eqs. (a) and (b), we have

$$8yz + (2\lambda / a^{2)}\, x = 0 \qquad \text{(d)}$$

$$8xz + (2\lambda / b^{2)}\, y = 0 \qquad \text{(e)}$$

$$8xy + (2\lambda / c^{2)}\, z = 0 \qquad \text{(f)}$$

Multiplying Eq. (d) by x, Eq. (e) by y, Eq. (f) by z, and adding them, we obtain

$$3U + 2\,\lambda\,(x^2/a^2 + y^2/b^2 + z^2/c^2) = 0 \qquad \text{(g)}$$

Example 2 (continued)

When substituting Eq. (a) into Eq. (g), the result is

$$3U + 2\lambda = 0$$

$$2\lambda = -3U \quad \text{or} \quad \lambda = -(3/2)\,U \qquad \text{(h)}$$

When we combine this with Eq. (d), we obtain

$$u\,[\,1 - (3/a^2)x^2\,] = 0 \qquad \text{(i)}$$

or,

$$x = a/\sqrt{3} \qquad \text{(j)}$$

Similarly, we find that $y = b/\sqrt{3}$, $z = c/\sqrt{3}$

Thus, the required maximum volume is $u = \dfrac{8\,abc}{3\,\sqrt{3}}$

Example 3: Heat Exchanger

A total of 300 lineal feet of tubes must be installed in a heat exchanger in order to provide the necessary heat-transfer surface area. The total dollar cost of the installation includes:

1. Cost of tubes, $700
2. Cost of shell = $25D^{2.5} L$
3. Cost of floor space for heat exchanger = $20DL$

The spacing of the tubes is such that 20 tubes will fit in a cross-sectional area of 1 ft^2 inside the shell. The optimization should determine the diameter D and the length L of the heat exchanger to minimize the purchase cost.

Example 3 (continued)

The objective function is made up of three costs:

$$C = 700 + 25\,D^{2.5}\,L + 20\,DL \qquad (1)$$

subject to the functional constraint

$$\pi(D^2/4)L\ (20\ \text{tubes/ft}^2) = 300\ \text{ft}$$

$$5\,\pi\,D^2\,L = 300$$

$$\varphi = L - 300\,/\,(5\,\pi\,D^2\,L) = 0 \qquad (2)$$

$$\frac{\partial C}{\partial D} = (25)(2.5)D^{1.5}L + 20\,L \qquad (3)$$

$$\lambda\frac{\partial \varphi}{\partial D} = -\,\lambda\,(300/5\pi)(-\,2D/D^4) = 60\,\lambda\,(2/\,\pi D^3)$$

Example 3 (continued)

$$\frac{\partial C}{\partial D} + \lambda\frac{\partial \varphi}{\partial D} = 62.5\ D^{1.5}\ L + 20\ L + 2\lambda\ (60/\pi D^3) \qquad (5)$$

$$\frac{\partial C}{\partial L} = (25\ D^{2.5} + 20\ D$$

$$\lambda\frac{\partial \varphi}{\partial L} = \lambda$$

$$\frac{\partial C}{\partial L} + \lambda\frac{\partial \varphi}{\partial L} = 25\ D^{2.5} + 20D + \lambda = 0 \qquad (6)$$

with the constraint $\varphi = L - 300/(5\pi D^{2)} = 0$ (7)

since there are only two independent variables (D, L)

Example 3 (continued)

From Eq. (7), $L = 60/\pi D^3$

From Eq. (6), $\lambda = -25\,D^{2.5} - 20\,D$

When substituting Eqs. (8) and (9) into Eq. (5), we have

$$62.5\,D^{1.5}(60/\pi D^2) + 20(60/\pi D^2) + 2(-25\,D^{2.5} - 20D)60/\pi D^2 = 0$$

Eq. (10) simplifies to:

$$62.5\,D^{1.5} + 20 - 50D^{1.5} - 40 = 0$$

or $D^{1.5} = 1.6$, $D = 1.37$ ft

Example 3 (continued)

When substituting Eq. (11) into Eq. (8), we have

$$L = 60/\pi(1.37)^2 = 10.2 \text{ ft}$$

The cost for the optimal design is

$$C = 700 + 25(1.37)^{2.5}(10.2) + 20(1.37)(10.2)$$

$$= \$1,540$$

References

- “Design and Analysis of Experiments” Montgomery
- “Numerical Methods for Engineers” Chapra and Canale

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